Math 142 Lecture 18 Notes

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1 Surfaces and Cellular Decomposition

1.1 Non-Hausdorff orbit space of a nice action

In Problem 2b on Homework 7, there was a problem that was incorrect based on the definitions we gave in class. The problem said "If X is an n-manifold with no boundary, and G acts nicely on X, then X/G is an n-manifold." It turns out that for X/G to be Hausdroff, you need additional conditions on the action.

Example 1.1. Here is a counterexample to the problem as it was written, where the orbit space X/G is not Hausdorff. Let $\mathbb{Z}$ act nicely on $X = \mathbb{R}^2 \setminus \{(0,0)\}$ by $f_n(x,y) = (2^n x, 2^{-n} y)$. Claim: In $X/\mathbb{Z}$, the images of $(1,0)$ and $(0,1)$ cannot be separated by open sets. The idea is that if you let $U_{(0,1)}$ be a small open ball around $(0,1)$ and apply f_1 repeatedly to $U_{(0,1)}$, the ball gets moved downward toward $y=0$ and stretched wider and wider. So the image of this ball will intersect any neighborhood of $U_{(1,0)}$, and the claim holds.

1.2 Surfaces

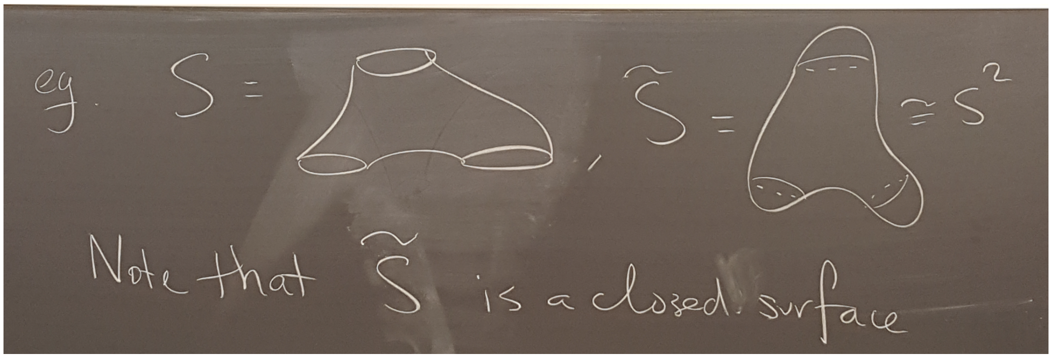
Definition 1.1. A *surface* is a 2-manifold (with or without boundary).

Proposition 1.1. *If S is a compact surface, then $\partial S \cong A_1 \amalg \cdots \amalg A_n$, where $A_i \cong S^1$ $\forall i$.*

Proof. From a proposition mentioned in class (and proved in HW7), we get that if S is a compact surface, then ∂S is a compact 1-manifold with no boundary. Then from our classification theorem, there is only one closed, connected 1-manifold, S^1. □

So let $\tilde{S} = S \cup_f (D_1 \amalg \cdots \amalg D_n)$, where $D_i \cong B^2$ $(= D^2)$ for all i and the $\mathrm{dom}(f) = \partial D_1 \amalg \cdots \amalg \partial D_n$ and $f|_{\partial D_i} : \partial D_i \to A_i$ is a homeomorphism. Note that $\tilde{S}$ is a closed surface.

Example 1.2. In the following image, S is called a "pair of pants."[1]

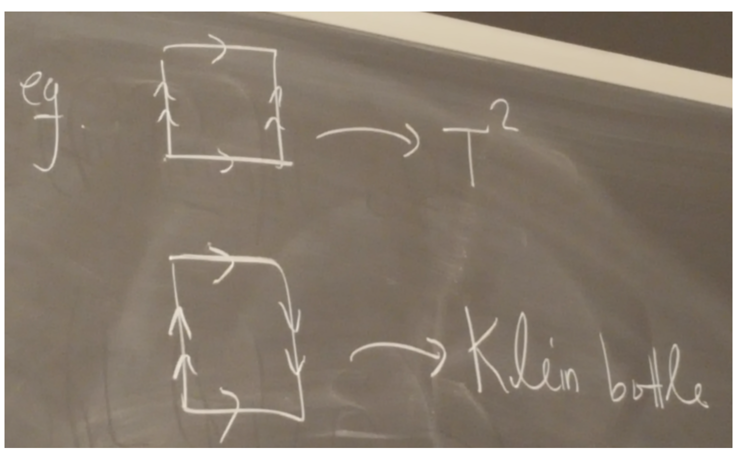


This means that to classify compact surfaces, we can restrict to closed surfaces.

Definition 1.2. A *cellular decomposition* of a closed surface S is a collection $\{P_i, \phi_i)\}$, where $P_i \subseteq \mathbb{R}^2$ is a filled-in polygon region (e.g. a filled in pentagon) and $\phi_i : P_i \to S$ such that

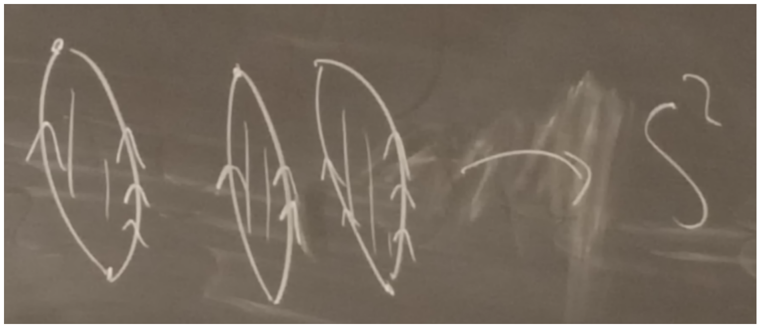
1. $\forall x \in S$, $x \in \phi_i(P_i)$ for some i.
2. $\phi_i|_{\text{int}(P_i)} : \text{int}(P_i) \to S$ is an embedding.
3. $\phi_i|_{\text{int}(e)} : \text{int}(e) \to S$ is an embedding for each edge $e \subseteq P_i$.
4. If $A_{i,j} = \phi_i(P_i) \cap \phi_j(P_j) \neq \varnothing$ for some $i \neq j$, then either $\phi_i^{-1}(A_{i,j})$ and $\phi_j^{-1}(A_{i,j})$ are entire edges of P_i and P_j, or A is a singleton with $\phi_i^{-1}(A_{i,j})$ and $\phi_j^{-1}(A_{i,j})$ being vertices.

Example 1.3. Our identification space drawings with the square each constitute a cellular decomposition with a single polygon.



[1] Aptly named.

Example 1.4. Here is a cellular decomposition of S^2 into three bi-gons.



1.3 Outline of the classification of 2-manifolds

Here is a fact we will not prove.

Theorem 1.1. *Every surface admits a cellular decomposition. If S is compact, then it admits a finite cellular decomposition.*

We will work

Theorem 1.2. *If S is a closed, connected surface, then*

$$S \cong S^2 \# \underbrace{T^2 \# \cdots \# T^2}_{n} \# \underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{m},$$

where n or m could be 0.

This is only half of a structure theorem. In Homework 8, we will prove the following fact.

Theorem 1.3.

$$T^2 \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2.$$

We will prove that S^2, $T^2 \# \cdots \# T^2$, and $\mathbb{R}P^2 \# \mathbb{R}P^2$ are all distinct, which will give us our classification.

Corollary 1.1 (Classification of 2-manifolds)**.** *If S is a closed, connected surface, then either*

1. $S \cong S^2$,
2. $S \cong T^2 \# \cdots \# T^2$, *or*
3. $S \cong \mathbb{R}P^2 \# \mathbb{R}P^2$.

1.4 Single-polygon cellular decomposition and words

We will use the following:

1. If $e_1 \subseteq P_1$ and $e_2 \subseteq P_2$ are edges such that $\phi_1(e_1) = \phi_2(e_2)$, then we can glue P_1 to P_2 along e_1 and e_2 to get a new polygon (P, ϕ) (where ϕ is ϕ_1 on P_1 and ϕ_2 on P_2) and replace $(P_1, \phi_1), (P_2, \phi_2)$ by (P, ϕ).

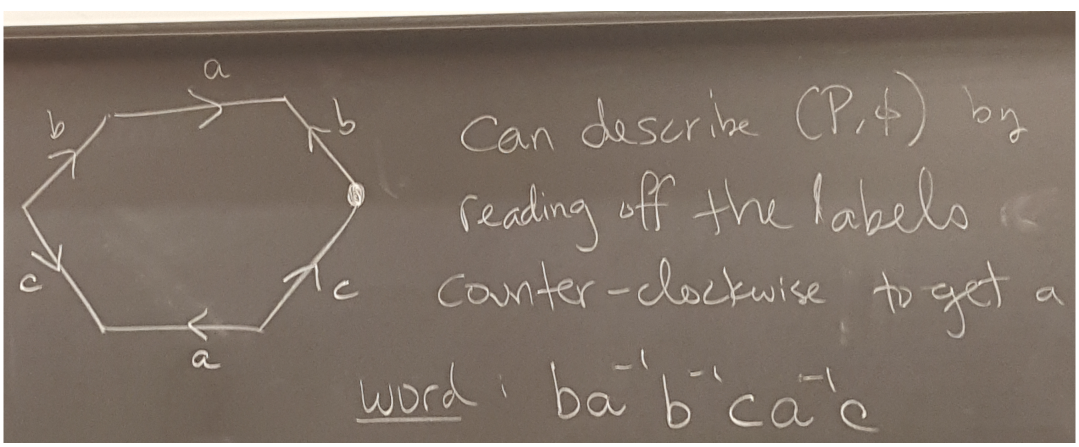
2. We can cut P_i into two pieces along a diagonal.

Lemma 1.1. *Suppose S is connected and closed. Then S has a cellular decomposition (P, ϕ) with a single polygon.*

Proof. S is compact, so choose a finite cellular decomposition $\{(P_i, \phi_i)\}$. Since S is connected and is a manifold, there exist (P_1, ϕ_1) and (P_2, ϕ_2) such that $\phi_1(e_1) = \phi_2(e_2)$ for some edges $e_1 \subseteq P_1$ and $e_2 \subseteq P_2$. Then we can glue P_1 and P_2 together along e_1 and e_2 and reduce the number of polygons in our decomposition. By repeating this process, we arrive at a single polygon. □

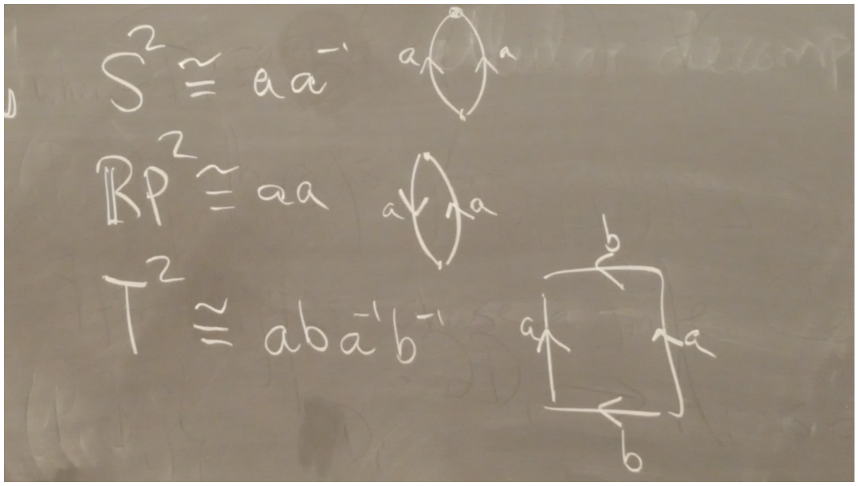
Given (P, ϕ), label the edges of P as follows: if $\phi(e_1) = \phi(e_2)$, label them the same, and put arrows indicating an orientation so that the arrows in S agree. We can describe (P, ϕ) by reading odd the labels counterclockwise to get a *word*, where the arrow gives a if the arrow goes counterclockwise and a^{-1} if the arrow goes counterclockwise. Any cyclic permutation of letters is equivalent.

Example 1.5.



We will write S as its word. So in the above example, $S \cong ba^{-1}b^{-1}ca^{-1}c$.

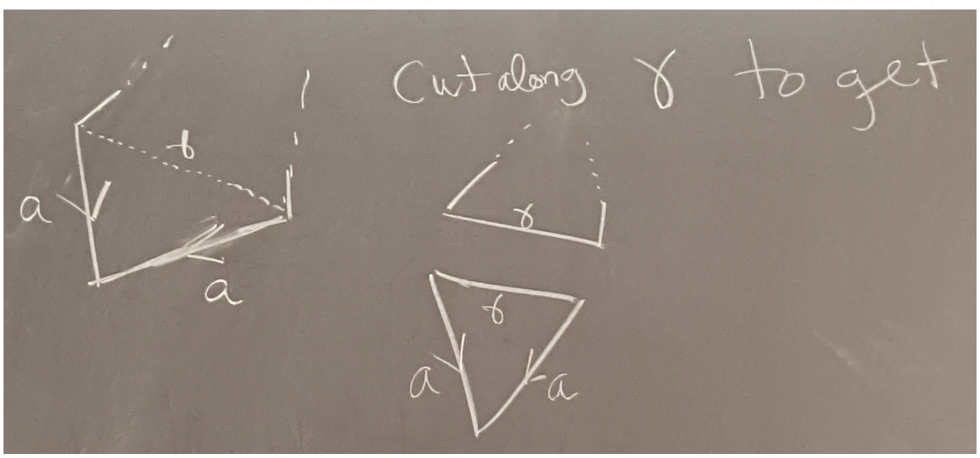
Example 1.6.



The following lemma says that adding a connecting a sphere to our surface essentially does nothing to it.

Lemma 1.2. *If $S \cong Xaa^{-1}$, and $X \neq \varnothing$, then $S \cong X$.*

Proof. We have the following picture. Cut along γ to get two polygons.



Note that $\phi(\gamma)$ is a closed loop in S, as the endpoints of γ are at vertices v_1, v_2 with $\gamma(v_1) = \gamma(v_2)$. Now glue a D^2 to each copy of γ. Notice that γ separates S into two disjoint connected components (since a only appears on one side of γ and no other letters). So $S \cong S' \# S''$. We now just need to show that one of these pieces is S^2; we will do this later. ☐